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ON THE FUNDAMENTAL PROBLEM OF THE DIFFERENTIAL CALCULUS.

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1. The fundamental problem of the Differential Calculus may be stated thus: Given the values of a function and its derivatives for a value a of the variable, to determine the values of the function and its derivatives for another value x of the variable. The general solution of this problem results in the Taylor-Lagrange formula for the expansion of the function in infinite series. The presentation of this all important theorem by the text books used in America for undergraduate students is wholly unsatisfactory. One need only refer to the treatise of Todhunter or Williamson in order to substantiate this statement. Either the formula is tentatively laid down to start with or the vicious assumption of the infinite series with undetermined coefficients is made and the series differentiated inexcusably in order to evaluate the coefficients; then subsequently, usually after applications to specific functions, Lagrange's theorem on the remainder is deduced and the so-called "cases of failure" commented upon.

2. The *fundamental principle* of the Calculus is the calculation of the function which is the limit of the finite difference ratio

$$\frac{f(x)-f(a)}{x-a},$$

as x converges to the limit a. This process is called differentiation, and the resulting function called the first derivative of $f(x)$ at a. Repeated application of this process results in the successive derivatives of $f(x)$ at a.

3. The *fundamental problem* of the Calculus is solved by repeated application of the process of differentiation to this same finite difference ratio

$$\frac{f(x)-f(a)}{x-a}.$$

Let $f(x)$ be a one-valued function of x which is finite and continuous throughout an interval $(\alpha\beta)$ of the variable. Let x and a be two values of the variable in the interval $(\alpha\beta)$. Let the first n derivatives of $f(x)$ be finite and determinate in the neighborhood of a. Apply Leibnitz formula for the differentiation of a product to the successive differentiation of the function

$$\frac{f(x)-f(a)}{x-a}$$

n times with respect to a. Whence, after multiplying both sides by $(x-a)^{n+1}/n!$, then results

$$f(x) - \overset{n}{\underset{0}{\Sigma}}{}_r \frac{(x-a)^r}{r!} f^r(a) = \frac{(x-a)^{n+1}}{n!} \left[\frac{d}{da}\right]^n \frac{f(x)-f(a)}{x-a}. \tag{1}$$

The member on the right represents the remainder at once in definite form, which can easily be thrown into Lagrange's form in the usual way.

Let z be a value of the variable in the interval (ax), and let

$$\varphi(z) = f(x) - \overset{n}{\underset{0}{\Sigma}}{}_r \frac{(x-z)^r}{r!} f^r(z) - \frac{(x-z)^{n+1}}{n!} \left[\frac{d}{da}\right]^n \frac{f(x)-f(a)}{x-a}.$$

If the first n derivatives of $f(x)$ are finite and continuous throughout (ax) the function $\varphi(z)$ will be also. We have $\varphi(x) = 0$, also $\varphi(a) = 0$ in virtue of (1). Therefore $\varphi'(z) = 0$ for some value u of z between x and a, and we have

$$\left[\frac{d}{da}\right]^n \frac{f(x)-f(a)}{x-a} = \frac{f^{n+1}(u)}{n+1}. \qquad a < u < x.$$

This completes the formula

$$f(x) = \overset{n}{\underset{0}{\Sigma}}{}_r \frac{(x-a)^r}{r!} f^r(a) + \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}(u). \tag{2}$$

If now the successive derivatives are determinate at a, and the $(n+1)$th derivative of $f(x)$ for all points between a and x is such that

$$\mathop{\mathcal{L}}_{n=\infty} \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}(u) = 0,$$

we have

$$f(x) = \overset{\infty}{\underset{0}{\Sigma}}{}_r \frac{(x-a)^r}{r!} f^r(a), \tag{3}$$

throughout the interval.

4. The process usually employed for determining Lagrange's theorem on the "remainder" in which it is assumed that the remainder after the $(n+1)$th term can be written, in general, in the form $(x-a)^m Q$ in which Q is assumed to be some function of only x and a is not to be approved on first reading without careful examination, if indeed it is a legitimate assumption under any circumstances. This assumption leads to the result

$$R_{n+1} = \frac{(1-\theta)^{n+1-m}}{m} \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}(u), \qquad 0 < \theta < 1$$

in which m is a function of u and must not be greater than $n+1$. The method of Art. 3 seems to be free from such objections inasmuch as (1) is an algebraical identity depending only on the process of differentiation of a function of finite form.

5. If (3) be true, then throughout the same interval, we must also have

$$f'(x) = \sum_1^\infty{}_r \frac{(x-a)^{r-1}}{(r-1)!} f^r(a) .$$

For, differentiating (1), we have

$$f'(x) = \sum_1^n{}_r \frac{(x-a)^{r-1}}{(r-1)!} f^r(a) + \frac{(x-a)^n}{n!}\left[(n+1)P + (x-a)\frac{dP}{dx}\right]$$

wherein

$$P = \left[\frac{d}{da}\right]^n \frac{f(x)-f(a)}{x-a} = \frac{f^{n+1}(u)}{n+1} .$$

Let

$$J(z) = f'(x) - \sum_1^n{}_r \frac{(x-z)^{r-1}}{(r-1)!} f^r(z) - \frac{(x-z)^n}{n!} Q ,$$

wherein

$$Q = (n+1)P + (x-a)\frac{dP}{dx} ,$$

and is independent of z.

As before, if the first n derivatives of $f(z)$ are finite and continuous throughout (ax), the function $J(z)$ will be also, for no higher derivatives are involved. We have $J(x) = 0$ and $J(a) = 0$, consequently $J'(v) = 0$ where v is some value of the variable in (ax). Hence

$$Q = f^{n+1}(v) ,$$

and we have

$$f'(x) = \sum_1^n{}_r \frac{(x-a)^{r-1}}{(r-1)!} f^r(a) + \frac{(x-a)^n}{n!} f^{n+1}(v) ,$$

and since

$$\frac{(x-a)^n}{n!} f^{n+1}(v) \text{ and } \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}(u)$$

vanish when $n = \infty$ under the same circumstances, we have

$$f'(x) = \sum_1^\infty{}_r \frac{(x-a)^{r-1}}{(r-1)!} f^r(a) ,$$

for the same values of the variable for which (3) holds. And in general,

$$f^m(x) = \sum_m^\infty{}_r \frac{(x-a)^{r-m}}{(r-m)!} f^r(a) ,$$

throughout the same interval for which (3) is true.

Thus the general problem is solved, the series (3) giving the value of the function at x in terms of the function and its derivatives at a. The successive derivatives of this series are shown to be the successive derivatives of the function at x.

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